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KO'P O'CHOVLI SOHADA O'ZGARUVCHAN ZICHLIK VA MANBA BILAN BERILGAN CHIZIQSIZ PARABOLIK TENGLAMA YECHIMLARINING XOSSALARI

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Annotatsiya. *Ushbu maqolada o'zgaruvchan zichlik va qo'shimcha manba bilan berilgan chiziqsiz reaksiya-diffuziya tipidagi tenglamaning turli yechim xossalari tadqiq qilingan. Energiya metodi yordamida ushbu tipdagi istalgan parabolik tenglamaning blow-up holatga kelishi ya'ni chekli vaqtda yechimning cheksizga qarab ketishi isbotlangan. Avtomodel tenglamaga keltirilib masalani yechish uchun eng yaqin boshlang'ich yaqinlashish-avtomodel yechim va uning noma'lum parametrlari aniqlangan.*

Kalit so'zlar. *Reaksiya-diffuziya, energiya metodi, blow-up, chiziqsiz tenglama, avtomodel yechim, Gyolder va Young tengsizliklari.*

СВОЙСТВА РЕШЕНИЙ НЕЛИНЕЙНЫХ ПАРАБОЛИЧЕСКИХ УРАВНЕНИЙ, ЗАДАВАЕМЫХ ПЕРЕМЕННОЙ ПЛОТНОСТЬЮ И ИСТОЧНИКОМ В МНОГОМЕРНОМ ПРОСТРАНСТВЕ

Аннотация. *В этой статье исследуются различные свойства решения нелинейного уравнения реакции-диффузии, заданного переменной плотностью и дополнительным источником. С помощью энергетического метода доказано, что любое параболическое уравнение такого типа имеет режим с обострением (blow-up) , что обозначает решение бывает бесконечным за конечное время. Ближайшим начальным приближением к решению задачи путем введения автомодели в уравнение является автомодельное решение и его неизвестные параметры.*

Ключевые слова. *Реакция-диффузия, энергетический метод, blow-up, нелинейное уравнение, автомодельное решение, неравенства Гельдера и Юнга.*

PROPERTIES OF SOLUTIONS OF NONLINEAR PARABOLIC EQUATIONS GIVEN BY VARIABLE DENSITY AND SOURCE

Annotation. *In this article, various solution properties of the nonlinear reaction-diffusion equation given by variable density and additional source are investigated. Using the energy method, it has been proved that any parabolic equation of this type will blow-up, which the solution will go to infinity at a finite time. The closest initial approximation to solve the problem by introducing the self-similar equation is the self-similar solution and its unknown parameters.*

Key words. *Reaction-diffusion, energy method, blow-up, nonlinear equation, self-similar solution, Holder and Young inequalities.*

KIRISH

Bizga quyidagi tenglama berilgan bo‘lsin

$$\rho_1(x)\frac{\partial u}{\partial t} = \nabla\left(\rho_2(x)u^{m-1}\left|\nabla u^k\right|^{p-2}\nabla u\right) + \rho_3(x)u^\beta \tag{1}$$

Bu yerda $u(x,t)$-qidirilayotgan yechim, ya’ni nomalum funksiya, $\rho_i = |x|^{n_i}, i = \overline{1,3}$ o‘zgaruvchan zichlik, u^β- manba yoki yutilish, $\rho_2(x)u^{m-1}\left|\nabla u^k\right|^{p-2}\nabla u$ -oqimni ifodalaydi. Ushbu tenglama orqali ko‘plab fizik, kimyoviy va biologik jarayonlar ifodalanib, shulardan eng ko‘p o‘rganilgani diffuziya, filtratsiya, populyatsiya jarayonlarini ifodalaydi. (1) tenglama ko‘p hollarda reaksiya-diffuziya jarayonini xarakterlab bu yerda $\frac{\partial u}{\partial t}$-diffuziya tezligini ifodalaydi. m, k, p, β -parametrlar oldindan berilgan bo‘lib, uning son qiymati qaralayotgan tenglama qanday jarayonni ifodalashiga bog‘liq.

MAVZUGA OID ADABIYOTLARNING TAHLILI

Ko‘pgina fizik kimyoviy, bioligik jarayonlar chiziqsiz parabolik tipdagi tenglamalar orqali ifodalanganligi tufayli ko‘plab olimlar ushbu jarayonni o‘rgangan. Xususan juda ko‘p maqolalar [1-3] chiziqsiz chegaraviy shartlar va Koshi masalasini o‘rganishga bag‘ishlangan. So‘nggi yillarda O‘zbekistonda M.M.Aripov [4-6], A.S.Matyakubov[7-8], A.Xaydarov [9-10] kabilar bunday chiziqsiz parabolik tipdagi tenglamalar orqali ifodalanuvchi turli masalalarning yechim xossalarini tadqiq etishgan.

TADQIQOT METODOLOGIYASI

Ushbu tenglamani quyidagi ko‘rinishda yozsak,

$$\rho_1 u_t - \rho_3 u^\beta = \nabla \rho_2 u^{m-1}\left|\nabla u^k\right|^{p-2}\nabla u + \rho_2 |k|^{p-2}\nabla\left(u^{m-1+(k-1)(p-2)}\nabla u^{p-2}\nabla u\right), \quad (2)$$

nomalumning darajasiga qarab uning chiziqsiz ekanligini ko‘rishimiz mumkin.

Energiya metodi.

Quyidagi funksiyani qaraymiz; $E(t) = \frac{1}{s}\int \rho_1(x)u^s(x,t)\psi_\varepsilon(x)$, bu yerda

$$\psi_\varepsilon(x) = A\varepsilon^N e^{-\varepsilon|x|}, \varepsilon > 0 \quad (3)$$

$I = \int_{R^N}\psi_\varepsilon(x)\,dx$ ni qaraymiz, bu yerda I -birlik operator. $\nabla\psi_\varepsilon = A\varepsilon^N(-\varepsilon)\frac{x}{|x|}e^{-\varepsilon} = -\varepsilon\frac{x}{|x|}\psi_\varepsilon$

$I = \int_{R^N}\psi_\varepsilon(x)dx = A\varepsilon^N\int_{R^N} e^{-\varepsilon|x|}dx \begin{cases} x = \frac{y}{\varepsilon} \\ dx = \frac{1}{\varepsilon}dy \end{cases} = A\int_{R^N} e^{-|y|dy}$ ekanligidan $A = \frac{1}{\int_{R^N} e^{-|x|dx}}$

$$E'(t) = \frac{1}{s}\cdot s\int_{R^N}\rho_1 u_t u^{s-1}\,\psi_\varepsilon dx$$

$$= \int_{R^N}\left[\nabla\left(\rho_2(x)u^{m-1}\left|\nabla u^k\right|^{p-2}\nabla u\right) + \rho_3 u^\beta\right]u^{s-1}\psi_\varepsilon dx$$

$$= \int_{R^N}\nabla\left(\rho_2(x)u^{m-1}\left|\nabla u^k\right|^{p-2}\nabla u\right)u^{s-1}\psi_\varepsilon dx + \int_{R^N}\rho_3\,u^{\beta+s-1}\psi_\varepsilon dx$$

$$= \int_{R^N}\rho_3\,u^{\beta+s-1}\psi_\varepsilon dx +$$

$$+\rho_2(x)u^{m-1}\left|\nabla u^k\right|^{p-2}\nabla u\cdot u^{s-1}\psi_\varepsilon\Big|_{R^N} - \int_{R^N}\left[(s-1)u^{s-2}\nabla u\psi_\varepsilon - \varepsilon\frac{x}{|x|}u^{s-1}\psi_\varepsilon\right]* k^{p-2}\rho_2 u^{m-1+(k-1)(p-2)}|\nabla u|^{p-2}\nabla u dx \quad (4)$$

$\rho_2(x)u^{m-1}\left|\nabla u^k\right|^{p-2}\nabla u\cdot u^{s-1}\psi_\varepsilon\Big|_{R^N} = 0$ ekanligini hisobga olib quyidagi tengsizlikni yoza olamiz

Young tengsizligidan foydalanib quyidagini yoza olamiz

$$\varepsilon\int_{R^N}u|\nabla u|^{p-2}\nabla u\psi_\varepsilon dx = \int_{R^N}\left(\varepsilon u\psi_\varepsilon^{\frac{1}{q'}}\right)\left(|\nabla u|^{p-2}\nabla u\psi_\varepsilon^{\frac{1}{p'}}\right)dx \le \frac{p-1}{p}\int_{R^N}|\nabla u|^p\psi_\varepsilon dx + \frac{\varepsilon^p}{p}\int_{R^N}u^p\psi_\varepsilon dx \quad (5)$$

Bu yerda $\frac{1}{q'} = \frac{1}{p}$va $\frac{1}{p'} = \frac{p-1}{p}$. Shunga ko‘ra quyidagi tengsizlik kelib chiqadi;

$$\varepsilon k^{p-2}\int_{R^N}\rho_2 u^{s+m-2+(k-1)(p-2)}|\nabla u|^{p-1}\,\nabla u\psi_\varepsilon dx$$

$$\le k^{p-2}\frac{(p-1)}{p}\int_{R^N}\rho_2 u^{s+m-3+(k-1)(p-2)}|\nabla u|^p\,\nabla u\psi_\varepsilon dx +$$

$$+\frac{\varepsilon^p k^{p-2}}{p}\int_{R^N}\rho_2 u^{s+m-3+k(p-2)+2-p+p}|\nabla u|^{p-1}\nabla u\psi_\varepsilon dx$$

Gyolders tengsizligidan foydalanib quyidagi tengsizlikni hosil qilamiz

$$E'(t)\geq\int_{R^N}\rho_3 u^{\beta+s-1}\psi_\varepsilon dx-\frac{\varepsilon^p k^{p-2}}{p}\int_{R^N}\rho_2 u^{s+m-1+k(p-1)}\psi_\varepsilon dx\leq$$

$$\leq\left(\int_{R^N}\rho_3 u^{\beta+s-1}\psi_\varepsilon dx\right)^{\frac{1}{p''}}\left(\int_{R^N}\rho_2{}^{q''}\rho_3{}^{-\frac{q''}{p''}}\psi_\varepsilon dx\right)^{\frac{1}{q''}}\left(\int_{R^N}\rho_2{}^{q''}\rho_3{}^{-\frac{q''}{p''}}\psi_\varepsilon dx\right)^{\frac{n}{q''}} \quad (6)$$

Bu yerda $p''=\frac{\beta+s-1}{s+m-1+k(p-2)}, q=\frac{\beta+s-1}{\beta+m-k(p-2)}$

$\nabla\psi_\varepsilon=A\varepsilon^N(-\varepsilon)\frac{x}{|x|}e^{-\varepsilon}=-\varepsilon\frac{x}{|x|}\psi_\varepsilon$ ekanligini hisobga olib $A\int_{R^N}|x|^{n_2q''-n_3\frac{q''}{p''}}e^{-|x|}dx=C_1>0$ deb belgilasak quyidagiga ega bo‘lamiz

$$E'(t)\geq\int_{R^N}\rho_3 u^{\beta+s-1}\psi_\varepsilon dx-C_1\varepsilon^{\frac{p+n_3\frac{1}{p'}-n_2k^{p-2}}{p}}\left(\int_{R^N}\rho_3 u^{\beta+s-1}\psi_\varepsilon dx\right)^{\frac{1}{p''}}$$

(7)

$T^*=\frac{(E(0))^{1-p}}{C_3}$ tenglamaning Life-Span holatini ifodalaydi, ya’ni ixtiyoriy $T\leq T^*$ uchun $E(t)\leq\frac{1}{\left((E(0))^{1-p'}-C_3t\right)^{\frac{2}{p'-1}}}$ tengsizlik o‘rinli bo‘ladi. Bu yerda $C_3=\frac{p_1-1}{2}\left(\frac{s}{C_2}\right)^{p_1}\varepsilon^{n_1p_1-n_3}>0$, $p_1=\frac{\beta+s-1}{s}$, $q_1=\frac{\beta+s-1}{\beta-1}$, $\beta>1$ va $C_2=\left(A\int|x|^{n_1q_1-n_3\frac{q_1}{p_1}}e^{-|x|}dx\right)^{\frac{1}{q_1}}>0$ ga teng

$E(t)\to\infty$ da $t\to T^*$. Bu yerdan tenglamaning istalgan parametrlaridan qatiy nazar blow-up holatga kelishi, ya’ni chekli vaqtda yechimning cheksizga intilishini ko‘rishimiz mumkin.

TAHLIL VA NATIJALAR

Berilgan sohani dekart fazosidan qutb koordinatalar sistemasiga o‘tkazib,

$$\sum_{i=1}^N\left(\frac{dN}{dx_i}\right)^2=\sum_{i=1}^N\left(\frac{x_i{}^2}{r^2}\right)=\frac{1}{r^2}\sum_{i=1}^N(x_i)=1;\text{ xossadan foydalanamiz}$$

$$\sum_{i=1}^N\left(\frac{d^2r}{dx_i{}^2}\right)=\sum_{i=1}^N\left(\frac{r^2-x_i{}^2}{r^3}\right)=\frac{1}{r}\sum_{i=1}^N 1-\frac{1}{r^3}\sum_{i=1}^N(x_i{}^2)=\frac{N}{r}-\frac{1}{r};$$

Shunda tenglamamiz quyidagi ko‘rinishga keladi

$$\nabla(|x|^{n_2}u^{m-1}\left|\nabla u^k\right|^{p-2}\nabla u)$$

$$=\sum_{i=1}^{N}\frac{\partial}{\partial x_i}(|x|^{n_2}u^{m-1}\left|\frac{\partial u^k}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i})=[r=|x|]$$

$$=\sum_{i=1}^{N}\frac{\partial}{\partial x_i}(r^{n_2}u^{m-1}\left|\frac{\partial u^k}{\partial r}\right|^{p-2}\frac{\partial u}{\partial r}\frac{\partial r}{\partial x_i}\left|\frac{\partial r}{\partial x_i}\right|^{p-2})=$$

$$\sum_{i=1}^{N}\frac{\partial}{\partial r}(r^{n_2}u^{m-1}\left|\frac{\partial u^k}{\partial r}\right|^{p-2}\frac{\partial u}{\partial r})\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_i}\left|\frac{\partial r}{\partial x_i}\right|^{p-2}$$

$$+r^{n_2}u^{n-1}\left|\frac{\partial u}{\partial r}\right|^{p-2}\frac{\partial u}{\partial r}\sum_{i=1}^{N}(\frac{d^2r}{dx_i^2}\left|\frac{d^2r}{dx_i^2}\right|\left|\frac{dr}{dx_i}\right|^{p-2}+\frac{\partial r}{\partial x_i}\frac{\partial}{\partial x_i}(\left|\frac{\partial r}{\partial x_i}\right|^{p-2}));$$

L fazoda norma kiritilish qoidasiga ko‘ra nomalum funksiya normasi quyidagiga teng

$$|\nabla u|=\sqrt{(\nabla u)^2}=\sqrt{(\sum_{i=1}^{N}\frac{\partial u}{\partial x_i})^2}=\sqrt{(\sum_{i=1}^{N}\frac{\partial u}{\partial r}\frac{\partial r}{\partial x_i})^2}=\sqrt{((\frac{\partial u}{\partial r})^2\sum_{i=1}^{N}(\frac{\partial r}{\partial x_i})^2)}$$

$$=\left|\frac{\partial u}{\partial r}\right|\sqrt{\sum_{i=1}^{N}(\frac{\partial r}{\partial x_i})^2}=\left|\frac{\partial u}{\partial r}\right|;$$

Bu yerda $r=|x|=\sqrt{\sum_{i=1}^{N}x_i^{\,2}}$; $\frac{\partial r}{\partial x_j}=\frac{x_j}{r}$; $\frac{\partial^2 r}{\partial x_j^{\,2}}=\frac{r^2-x_j^{\,2}}{r^3}$;

$\sum_{i=1}^{N}(\frac{\partial r}{\partial x_i})^2=\frac{1}{r^2}\sum_{i=1}^{N}x_i^{\,2}=1;$ va $\sum_{i=1}^{N}\frac{\partial^2 r}{\partial x_j^{\,2}}=\frac{1}{r}\sum_{i=1}^{N}1-\frac{1}{r^2}\sum_{i=1}^{N}x_i^{\,2}=\frac{N-1}{r};$

ekanligidan foydalanib yozsak

$$\nabla(\rho_2(x)u^{m-1}\left|\nabla u^k\right|^{p-2}\nabla u)=k^{p-2}\nabla(\rho_2(x)u^{m-1+(k+1)(p-2)}\left|\nabla u\right|^{p-2}\nabla u)=k^{p-2}\sum_{i=1}^{N}\frac{\partial}{\partial x_i}(r^{n_2}u^{m-1+(k+1)(p-2)}\left|\frac{\partial u}{\partial r}\right|^{p-2}\frac{\partial u}{\partial r})(\frac{\partial r}{\partial x_i})^2+k^{p-2}r^{n_2}u^{m-1+(k+1)(p-2)}\left|\frac{\partial u}{\partial r}\right|^{p-2}\frac{\partial u}{\partial r}\sum_{i=1}^{N}\frac{\partial^2 r}{\partial x_i^2}=$$

$$k^{p-2}r^{n_2}u^{m-1+(k+1)(p-2)}\left|\frac{\partial u}{\partial r}\right|^{p-2}\frac{\partial u}{\partial r}\sum_{i=1}^{N}(\frac{\partial r}{\partial x_i})^2+$$

$$k^{p-2}r^{n_2}u^{m-1+(k+1)(p-2)}\left|\frac{\partial u}{\partial r}\right|^{p-2}\frac{\partial u}{\partial r}\frac{N-1}{r}=$$

$$r^{n_2}u^{m-1}\left|\frac{\partial u^k}{\partial r}\right|^{p-2}\frac{\partial u}{\partial r}+\frac{N-1}{r}r^{n_2}u^{m-1}\left|\frac{\partial u^k}{\partial r}\right|^{p-2}\frac{\partial u}{\partial r}=r^{1-N}(r^{1-N}r^{n_2}u^{m-1}\left|\frac{\partial u^k}{\partial r}\right|^{R_2}\frac{\partial u}{\partial r}+$$

$$(N-1)r^{N-1}r^{n_2}u^{m-1}\left|\frac{\partial u^k}{\partial r}\right|^{R_2}\frac{\partial u}{\partial r})=r^{1-N}\frac{\partial}{\partial r}(r^{N-1+n_2}u^{m-1}\left|\frac{\partial u^k}{\partial r}\right|^{p-2}\frac{\partial u}{\partial r});\quad(8)$$

Bu yerda $u(t,r)=u(t,\varphi(r))$, $(\varphi')^p r^{n_2-n_1}=\frac{p}{n_1-n_2};\varphi'=r$, $\varphi(r)=\{\frac{pr^{\frac{p+n_1-n_2}{p}}}{p+n_1-n_2};n_2\neq n_1+p\}$,

$\{ln(n);n_2=n_1+p\}$ ga teng

$$r^{-n_1+1-N}((\varphi')^{p-1}r^{N-1+n_2})'=r^{1-N-n_1}(r^{N-1+n_2+n_1-n_2-\frac{n_1-n_2}{p}})$$
$$=\frac{p^2(N-1+n_1)-n_1+n_2}{p}\frac{1}{2^{\frac{p+n_1-n_2}{p}}}$$
$$=\frac{p(N-1+n_1)-n_1+n_2}{p}\frac{1}{\varphi\frac{p+n_1-n_2}{p}}=\frac{S-1}{\varphi};$$

$S=1+\frac{p(N-1+n_1)-n_1+n_2}{p+n_1-n_2}=\frac{p(N+n_1)}{p+n_1-n_2}$; deb olsak tenglama quyidagi ko‘rinishga keladi

$$\frac{\partial u}{\partial t}=\varphi^{1-S}\frac{\partial}{\partial\varphi}(\varphi^{S-1}u^{m-1}\left|\frac{\partial u^k}{\partial\varphi}\right|^{p-2}\frac{\partial u}{\partial\varphi})+r^{n_3-n_1}u^\beta;\quad(9)$$

Bu yerda $r^{n_3-n_1}=(\varphi\frac{p+n_1-n_2}{p})^{\frac{p(n_2-n_1)}{p+n_1-n_2}};r$ -qutb koordinatalar sistemasida radius, φ -burchak

XULOSA

Ushbu maqola orqali (1) ko‘rinishda aniqlangan parabolik tipdagi istalgan, o‘zgaruvchan zichlik va qo‘shimcha manba bilan berilgan chiziqsiz reaksiya-diffuziya tenglamaning blow up holatga kelishi isbotlab ko‘rsatilgan. Tadqiqot natijalari orqali ushbu tipdagi tenglamalar uchun avtomodel tenglama (11) va boshlang‘ich yaqinlashish uchun avtomodel yechim (10) ishlab chiqildi va uning global yechim bo‘lish sharti ko‘rsatildi.

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