LOCAL DERIVATIONS ON SOLVABLE LIE ALGEBRAS WITH A FILIFORM NILRADICAL

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Summary. *The present paper is devoted to the description of local derivations on rank one or zero solvable Lie algebras with the filiform nilradical. Firstly, we shall consider local derivations of rank one solvable Lie algebra* W_n^+ *and* R_n^+ *with the filiform nilradical* W_n *and* R_n*. Namely, we find a general form of local derivations in the case of solvable Lie algebras whose nilradical is the Witt algebra* W_n *and special filiform algebra* R_n*. In particular, in these cases the spaces* $\mathrm{LocDer}(R)$ *are Lie algebras. Finally, we shall find a general form of rank zero solvable Lie algebra with a metabelian filifrom nilradical.*

Key words. *Lie algebra, derivation, local derivation*

Rezyume. *Ushbu maqolada filiform nilradikal, rangi birga yoki nolga teng Li algebralarida lokal differentsiallashlarning tavsifiga bag'ishlangan. Biz* W_n^+ *va* R_n^+ *filiform nilradikalli rangi birga* W_n *va* R_n *ga teng yechimli Li algebrasining lokal differentsiallashlari ko'rib chiqilgan.* W_n *Witt algebrasi va maxsus* R_n *filiform algebrasi bo'lgan Li algebralarida lokal differentsiallashlarining umumiy ko'rinishi topilgan. Xususan bu holda* $\mathrm{LocDer}(R)$ *Li algebralaridir. Va nihoyat, rangi nolga teng metabel filiform nilradikalli yechiluvchan Li algebrasining umumiy ko'rinishi topilgan.*

Kalit so'zlar. *Li algebra, differentsiallash, lokal differentsiallash*

Резюме. *Настоящая работа посвящена описанию локальных дифференцирований на разрешимых алгебрах Ли ранга один или нуль с филиформным нильрадикалом. Рассматриваются локальные дифференцирования разрешимой алгебры Ли ранга один* W_n^+ *и* R_n^+ *с филиформный нильрадикалом* W_n *и* R_n. *А именно, мы находим общий вид локальных дифференцирований в случае разрешимых алгебр Ли, нильрадикалом которых является алгебра Витта* W_n *и специальной филиформной алгебры* R_n. *В частности, в этих случаях пространства* $\mathrm{LocDer}(\mathrm{R})$ *являются алгебрами Ли. Наконец, мы найдем общий вид разрешимой алгебры Ли ранга нуль с метабелевым филиформны нильрадикалом.*

Ключевые слова. *Алгебра Ли, дифференцирование, локальные дифференцирование.*

INTRODUCTION

Let A be an algebra (not necessary associative). Recall that a linear mapping $D:\mathrm{A}\to\mathrm{A}$ is said to be a derivation, if $D(xy)=D(x)y+xD(y)$ for all $x,y\in\mathrm{A}$. A linear mapping Δ is said to be a local derivation, if for every $x\in\mathrm{A}$ there exists a derivation D_x on A (depending on x) such that $\Delta(x)=D_x(x)$. The definition of local automorphism is similar [9]. These notions were introduced and investigated independently by R.V. Kadison [8] and D.R. Larson and A.R. Sourour [9]. The above papers gave rise to a series of works devoted to the description of mappings which are close to automorphisms and derivations of C^*-algebras and operator algebras. In [9] D.R. Larson and A.R. Sourour proved that if $\mathrm{A}=B(X)$, the algebra of all bounded linear operators on a Banach space X, then every invertible local automorphism of A is an automorphism. Thus automorphisms on $B(X)$ are completely determined by their local actions. In [4, Lemma 4] it was shown that the set of all local automorphisms $\mathrm{LocAut}(\mathrm{A})$ of an algebra A form a multiplicative group.

In [2] Sh.A. Ayupov and the first author have proved that every local derivation on semi-simple Lie algebra over an algebraically closed field of characteristic zero is a derivation and gave examples of finite-dimensional nilpotent Lie algebras with local derivations which are not derivations. In [15] the authors studied local derivatoions of standard Borel subalgebras of a finite-dimensional simple Lie algebra over an algebraically closed field of characteristic 0 and proved that every local derivation of such algebras is a derivation. In [3] have shown that in the class of solvable Lie algebras there exist algebras which admit local derivations which are not ordinary derivation and also algebras for which every local derivation is a derivation. The authors found necessary and sufficient conditions under which any local derivation of solvable Lie algebras with abelian nilradical and one-dimensional complementary space is a derivation. For the structure of solvable algebras with a given nilradical see [1, 7, 12].

2. Local derivations on solvable Lie algebras with a filiform nilradical

2.1 Solvable and filiform Lie algebras.

Let L be a Lie algebra. Consider the following central lower and derived sequences:

$$\mathrm{L}^1 = \mathrm{L}, \quad \mathrm{L}^{k+1} = [\mathrm{L}^k, \mathrm{L}], \qquad k \geq 1,$$
$$\mathrm{L}^{[1]} = \mathrm{L}, \quad L^{[s+1]} = [L^{[s]}, \mathrm{L}^{[s]}], \quad s \geq 1.$$

A Lie algebra L is called nilpotent (respectively, solvable), if there exists $p \in \mathrm{N}$ such that $\mathrm{L}^p = 0$ (respectively, $\mathrm{L}^{[p]} = 0$). The smallest integer k such that $\mathrm{L}^k = 0$ is called the nilindex (or the nilpotency class) of L.

A Lie algebra L is called filiform, if $\dim \mathrm{L}^k = n-k-1$ for $1 \leq k \leq n-1$. Note that the filiform Lie algebras have the maximal possible nilindex, $n-1$. These algebras are the "least" nilpotent.

Any Lie algebra L contains a unique maximal solvable (resp. nilpotent) ideal, called the radical (resp. nilradical) of the algebra (see [5, 6, 10, 14]).

Let L be a nilpotent Lie algebra and let $\mathrm{Der}(\mathrm{L})$ be the Lie algebra of all derivations of L. There is maximal abelian subalgebras of $\mathrm{Der}(\mathrm{L})$, consisting of semisimple elements. By a theorem of Mostow, these algebras are conjugate under the action of the group of inner automorphism of L. These abelian algebras are called maximal torus of derivations and the dimension of the maximal torus of $\mathrm{Der}(\mathrm{L})$ is an invariant of L. It is called rank of L [7].

It is known [7, 11] that the rank of solvable algebra with a filiform nilradical is at most 2. There are only two types of filiform Lie algebras of rank 2 and three types of filiform Lie algebras of rank 1 [7].

2.2 Local derivations of W_n^+

Let $n \geq 5$ and let W_n^+ be the $(n+1)$-dimensional solvable Lie algebra with a basis $\{e_0, e_1, e_2, \ldots, e_n\}$ such that

$$[e_i, e_j] = (j-i)e_{i+j}, \quad 0 \leq i, j \leq n, i+j \leq n.$$

Note that $\mathrm{W}_n = \left[\mathrm{W}_n^+, \mathrm{W}_n^+\right] = span\{e_1, \ldots, e_n\}$ is the n-dimesional filiform Lie algebra which called a Witt algebra. By [1, Theorem 4.1] all derivations of W_n^+ are inner.

For $i, j \in \{0, 1, \ldots, n\}$ let $E_{i,j}$ denote the linear mapping on W_n^+ defined on basis elements as follows

$$E_{i,j}(e_k) = \delta_{jk} e_i, k = 0, \ldots, n,$$

where δ_{jk} is the Kronecker delta. Set

$$\mathrm{P}Loc(\mathrm{W}_n^+) = span\left\{E_{i,j} : 0 \leq j \leq m-1, 2j+1 \leq i \leq n\right\},$$

if $n = 2m$ is even,

$$\mathrm{P}Loc(\mathrm{W}_n^+) = span\left\{E_{i,j}, E_{n,k} : 0 \leq j \leq m, 2j+1 \leq i \leq n, m+1 \leq k \leq n\right\},$$

if $n = 2m+1$ is odd.

Theorem 2.1. *Any local derivation* Δ *on* W_n^+ *is uniquely represented in the form*

$$\Delta = ad(a) + \overline{\Delta},$$

where $a \in span\left\{e_0, e_1, \ldots, e_{\left[\frac{n}{2}\right]-1}\right\}$ and $\overline{\Delta} \in PLoc(W_n^+)$, $[t]$ is the integer part of the real number t. Moreover, the space $LocDer(W_n^+)$ equipped with a Lie bracket is a Lie algebra and $PLoc(W_n^+)$ its ideal.

2.3 Local derivations of R_n^+

Let $n \geq 5$ and let R_n^+ be a $(n+1)$-dimensional solvable Lie algebra with a basis $\{e_0, e_1, \ldots, e_n\}$ such that

$$[e_0, e_i] = ie_i, \qquad 1 \leq j \leq n, e_1, e_i] = e_{i+1},$$
$$2 \leq i \leq n-1, e_2, e_i] = e_{i+2}, \quad 3 \leq i \leq n-2.$$

Note that by [1, Theorem 4.1] all derivations of R_n^+ are inner.

Denote

$$PLoc(R_n^+) = span\left\{E_{i,j} : 0 \leq j \leq 2, 2j+1 \leq i \leq n\right\}.$$

Theorem 2. *Any local derivation* Δ *on* R_n^+ *is uniquely represented in the form*

$$\Delta = ad(a) + \overline{\Delta},$$

where $a \in span\{e_0, e_1, e_2\}$ and $\overline{\Delta} \in PLoc(R_n^+)$. Moreover, the space $LocDer(R_n^+)$ equipped with a Lie bracket is a Lie algebra and $PLoc(R_n^+)$ its ideal.

Further we obtain description of the space of all local derivations of rank zero solvable Lie algebras with filiform nilradical, namely with a so-called matabelian filifrom Lie radical.

A non-abelian Lie algebra L is called a metabelian, if $L^{[2]} = 0$.

We shall consider a metabelian filiform Lie algebra L of dimension $n \geq 7$ with a basis $\{e_1, \ldots, e_n\}$ such that

$$[e_1, e_i] = e_{i+1}, \qquad 2 \le i \le n-1, e_2, e_i] = e_{i+2} + e_{i+2},$$
$$3 \le i \le n-2, e_2, e_{n-2}] = e_n.$$

In this subsection L is the Lie algebra with the above basis. By [13, Proposition 3.2.5], a derivation D on L has the strictly lower triangular matrix $(d_{i,j})$ such that:

$$\begin{aligned} &d_{2,1} = 0, \\ &d_{i+1,i} = d_{3,2}, && 3 \le i < n; \\ &d_{ij} = d_{i-j+2,2} - d_{i-j+1,1} - d_{i-j,1} && 3 \le j < i-1 < n. \end{aligned}$$

Note that the numbers $d_{i,j}$ ($3 \le i, j \le n$) comlpetely determined by $d_{k,1}, d_{k,2}$ ($k = 3, \ldots, n$) and the space of all derivations $\mathrm{Der(L)}$ has the dimension $2n-4$.

Theorem 3. *A linear mapping Δ on L is a local derivation if and only if it has the strictly lower triangular matrix $(\delta_{i,j})$ with $\delta_{2,1} = 0$.*

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