A NONLOCAL PROBLEM WITH BITSADZE-SAMARSKII CONDITIONS ON CHARACTERISTICS OF A DIFFERENT FAMILY FOR A PARABOLIC-HYPERBOLIC EQUATION

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Abstract. *In this paper, we study the second-order differential invariants of submersions with respect to the group of conformal transformations Euclidian spaces. In particular, it is proved that the ratio of principal surface curvatures is a second-order differential invariant with respect to the group of conformal transformations.*

Keywords: *Conformal transformation, differential invariants, submersion, vector field.*

MSC (2010): 53C12, 57R25, 57R35

1 Statement of the problem

We consider the equation

$$0 = Lu \equiv \begin{cases} u_{xx} - u_y, (x,y) \in D_1, \\ u_{xx} - u_{yy}, (x,y) \in D_2 \bigcup D_3 \end{cases} \quad (1.1)$$

,

where D_1 is one connected domain bounded by the segments AB, BB_0, B_0A_0, A_0A on the lines $y = 0$, $x = 1$, $y = h$, $x = 0$, respectively; D_2 is a characteristic triangle bounded by the segment AB of axis Ox and with the characteristics $AC_1 : x + y = 0$, $BC_2 : x - y = 1$ of equation (1) issuing from the points $A(0,0)$ and $B(1,0)$, intersecting at a point $C_1\left(\frac{1}{2};\ -\frac{1}{2}\right)$; D_3 is the characteristic triangle also, bounded by the segment AA_0 of axis Oy and with two characteristics $AC_2 : x + y = 0$, $A_0C_2 : y - x = 1$ of equation (1.1) issuing from the points $A(0,0)$ and $A_0(0,h)$, intersecting at a point $C_2\left(-\frac{1}{2};\ \frac{1}{2}\right)$.

We introduce the notations: $J \equiv AB = \{(x,y) : 0 < x < 1, y = 0\}$,

$I \equiv AA_0 = \{(x,y) : x = 0, 0 < y < h\}, D_1 = D \setminus \{x > 0, y > 0\}$,

$D_2 = D \setminus \{x > 0, y < 0\}, D_3 = D \setminus \{x < 0, y > 0\}, D = D_1 \cup D_2 \cup D_3 \cup J \cup I$, $I_1 = \{(x,y) : x = 0, 0 < y < k_2\}, I_2 = \{(x,y) : x = 0, k_2 < y < 1\}, k_2 \in I$,

$J_1 = \{(x,y) : 0 < x < k_1, y = 0\}, J_2 = \{(x,y) : k_1 < x < 1, y = 0\}, k_1 \in J$.

Let $P_1(P_2)$ and $Q_1(Q_2)$ denote, respectively, the points of intersection of the characteristics $AC_1(AC_2)$ and $BC_1(DC_2)$ with characteristics coming from points $E_1(k_1, 0) \in J(E_2(0,k_2) \in I)$,

$$\theta_1(x) = (x/2\,;\ -x/2\,), \theta_1^*(x) = ((x+k_1)/2\,;\ (k_1-x)/2\,), \qquad (1.2)$$

$$\theta_2(y) = (-y/2\,;\ y/2\,), \theta_2^*(y) = ((k_2-y)/2\,;\ (k_2+y)/2\,) \qquad (1.3)$$

$\theta_1(x)(\theta_2(y))$is the point of intersection of the characteristic AC_1 (AC_2) with a characteristic emerging from a point $M_1(x,0)\left(\tilde{M}_1(0,y)\right)$,

$(x,0) \in J_1\,((0,y) \in I_1)$, $\theta_1^*(x)\,(\theta_2^*(y))$ is the point of intersection of a $\left(\ E_1Q_1\,(E_2Q_2)\ \right)$ characteristicwith a characteristic emerging from a point $M_2(x,0)\ \tilde{M}_2(0,y)(x,0) \in J_2\,((0,y) \in I_2)$.

The present paper is devoted to the investigation of the problem with Bitsadze-Samarskii conditions (see [1]) on characteristics AP_j and characteristics AC_j, E_jQ_j $(j = 1,2)$ as one family.

BS-Problem. To find a function $u(x,y)$ in the domain D with the following properties:

1) $u(x,y) \in C(\bar{D})$;

2) $u(x,y) \in C^{2,1}_{x,y}(D_1 \cup AB \cup A_0B_0) \cap C^{2,2}_{x,y}(D_j \setminus (E_jP_j \cup E_jQ_j))$, satisfies equation (1) in the domains D_1 and $D_j \setminus (E_jP_j \cup E_jQ_j)$, $(j = 2,3)$;

3) $u_y \in C(D_1 \cup J_1 \cup J_2) \cap C(D_2 \cup J_1 \cup J_2)$ and on the intervals $J_j (j = 1,2)$ takes place gluing condition:

$$\lim_{y\to -0} u_y(x,y) = \lim_{y\to +0} u_y(x,y), (x,0) \in J_1 \bigcup J_2, \qquad (1.4)$$

5) $u(x,y)$ satisfies the boundary conditions

$$u|_{x=1} = \phi_1(y), 0 \le y \le h, \quad (1.5)$$

$$a_1(x)u[\theta_1(x)] + b_1(x)u(x,0) = c_1(x), (x,0) \in \bar{J}_1, \quad (1.6)$$

$$a_2(y)u[\theta_2(y)] + b_2(y)u(0,y) = c_2(y), (0,y) \in \bar{I}_1, \quad (1.7)$$

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$$u[\theta_1(x)] = \mu_1\, u[\theta_1^*(x)] + \delta_1(x), \quad (x,0) \in \bar{J}_2 \quad (1.8)$$

$$u[\theta_2(y)] = \mu_2\, u[\theta_2^*(y)] + \delta_2(y), \quad (0,y) \in \bar{I}_2, \quad (1.9)$$

where $\phi_1(y)$, $\delta_j(t)$, $a_j(t)$, $b_j(t)$, $c_j(t)$ $(j = 1,2)$ are given functions, and

$\mu_j \neq 1, c_j(k_j) = a_j(k_j)\delta_j(k_j) (j = 1,2), c_1(0) = c_2(0) = 0$, (1.10)

$a^2_j(t) + b^2_j(t) \neq 0, a_j(t) + 2b_j(t) > 0, \forall t \in [0,k_j]$, (1.11)

$\phi_1(y) \in C[0,h] \cap C^1(0,h), \delta_1(x) \in C^1(\bar{J}_2) \cap C^3(J_2), \delta_2(y) \in C^1(\bar{I}_2) \cap C^3(I_2)$, (1.12)

$a_j(t), b_j(t), c_j(t), \in C[0, k_j] \cap C^2(0, k_j), (j = 1,2)$. (1.13)

Notice, that

- Conditions (1.6) and (1.7) are Bitsadze - Samarskii conditions on the characteristics AP_j.

- Conditions (1.8) and (1.9) are mixing condition, where the non-local condition point wise links the values of the desired solution to the parallel characteristics AC_j and E_jQ_j $(j = 1,2)$.

Well known, that the analogs of the Tricomi problem for equation (1) have been studied in [3] - [5]. The *BS*-problem for equation (1.1) has not previously been investigated.

2 The main functional relations

In the study of the BS-problem, an important role is played functional relations between $\nu_1(x)(\nu_2(y))$ and $\tau_1(x)(\tau_2(y))$ from the parabolic and hyperbolic parts of the domain D, where

$$u(x,0) = \tau_1(x), \quad (x,0) \in \bar{J}, \lim_{y\to 0} u_y(x,y) = \nu_1(x), \quad (x,0) \in J, \quad (2.1)$$

$$u(0,y)=\tau_2(y), \quad (0,y)\in \bar{I}, \lim_{x\to 0} u_x(x,y)=\nu_2(y), \quad (0,y)\in I. \quad (2.2)$$

As we know [6], the solution of the Cauchy problem with initial conditions (2.1) for equation (1.1) in the domain D_2 has the form:

$$u(x,y)=\frac{1}{2}[\tau_1(x+y)+\tau_1(x-y)]+\frac{1}{2}\int_{x-y}^{x+y}\nu_1(t)\,dt. \quad (2.3)$$

By (1.2) from (2.3) we obtain

$$u[\theta_1(x)]=u\left[\frac{x}{2},-\frac{x}{2}\right]=\frac{1}{2}[\tau_1(0)+\tau_1(x)]+\frac{1}{2}\int_x^0\nu_1(t)dt, \quad (2.4)$$

$$u[\theta_1^*(x)]=u\left[\frac{x+k_1}{2},\ \frac{k_1-x}{2}\right]=\frac{1}{2}[\tau_1(k_1)+\tau_1(x)]+\frac{1}{2}\int_x^{k_1}\nu_1(t)dt. \quad (2.5)$$

By (1.10), (2.1), (2.2) from (1.6), (1.7), (1.8) and (1.9) it follows that

$$\tau_1(0)=\tau_2(0)=0, \tau(k_1)=\tau(k_2)=0. \quad (2.6)$$

Substituting (2.4), (2.5) into (1.6) and (1.8), taking (2.1) and (2.6) into account, respectively, we have

$$[a_1(x)+2b_1(x)]\,\tau_1(x)-\int_0^x\nu_1(t)dt=2c_1(x), \quad (x,0)\in\bar{J}_1, \quad (2.7)$$

and

$$(1-\mu_1)\tau_1(x)+\int_x^0\nu_1(t)dt=\mu_1\int_x^{k_1}\nu_1(t)dt+2\delta_1(x),(x,0)\in\bar{J}_2. \quad (2.8)$$

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Differentiating (2.7) and (2.8) with x respect to $\mu_1 6= 1$, respectively, we obtain the functional relation between $\tau_1(x)$ and $\nu_1(x)$, brought from the domain D_2 on the J_1 and J_2, which have the forms

$$\nu_1(x)-d_1(x)\tau_1'(x)-d'_1(x)\tau_1(x)=-2c'_1(x),(x,0)\in J_1 \quad (2.9)$$

and

$$\nu_1(x)-\tau_1^0(x)=-2\delta^0_1(x)/(1-\mu_1), (x,0)\in J_2, \quad (2.10)$$

where $d_1(x) = a_1(x) + 2b_1(x)$.

Similarly, using the solution

$$u\left(x;y\right)=\frac{1}{2}\left[\tau_2\left(x+y\right)+\tau_2\left(y-x\right)\right]-\frac{1}{2}\int\limits_{x+y}^{y-x}\nu_2\left(t\right)dt \qquad (2.11)$$

of the Cauchy problems (see [6] and [7]) with the initial date (2.2) for equation (1.1) in the domain considering (2.10), (1.7) and (1.9) we obtain the functional relation between $\tau_2(y)$ and $\nu_2(y)$, brought from the domainD_3 on the I_1 and I_2:

$$\nu_2\left(y\right)-d_2(y)\tau_2'\left(y\right)-d'_2(y)\tau_2\left(y\right)=-2c'_2\left(y\right),(0,y)\in I_1 \qquad (2.12)$$

and

$$\nu_2\left(y\right)-\tau_2'\left(y\right)=-2\delta'_2(y)/(1-\mu_2)\,,\quad (0,y)\in I_2 \qquad (2.13)$$

respectively, where $d_2(y) = a_2(y) + 2b_2(y)$.

According to the conditions of the BS-problem, passing to the limit as $y \to +0$in equation (1.1), we obtain the functional relation between $\tau_1(x)$and $\nu_1(x)$, brought from the domain D_1 on J:

$$\tau^{00}{}_1(x) = \nu_1(x),\ (x,0) \in J. \qquad (2.14)$$

Solution of the first boundary value problem with conditions $u(x,+0) = \tau_1(x)$, $(x,0) \in J$, $u^-(+0,y) = \tau_2(y)$, $(0,y) \in I^-$ and (1.5) for equation (1.1) in domain D_1 has the form [8], [9]:

$$u\left(x,y\right)=\int\limits_0^y G_\xi\left(x,y;0,\eta\right)\tau_2\left(\eta\right)d\eta+\int\limits_0^y G_\xi\left(x,y;1,\eta\right)\varphi_1\left(\eta\right)d\eta+$$

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$$+\ G(x,y;\xi,0)\tau_1(\xi)d\xi, \qquad (2.15)$$

0

where

$G(x,y;\xi,\eta) = \frac{1}{2\sqrt{\pi(y-\eta)}}\sum\limits_{n=-\infty}^{+\infty}\left\{e^{-\frac{(x-\xi+2n)^2}{4(y-\eta)}}-e^{-\frac{(x+\xi+2n)^2}{4(y-\eta)}}\right\}$ is

Green's function of the first boundary value problem for the equation $u_{xx} - u_y = 0$.

Differentiating (2.15) with respect to *x*, we obtain

$$u_x(x,y)=\int_0^y G_{\xi x}(x,y;\,0,\eta)\tau_2(\eta)d\eta+\int_0^y G_{\xi x}(x,y;\,1,\eta)\tau_3(\eta)d\eta+$$

$$+\int_0^1 G_x(x,y;\,\xi,0)\tau_1(\xi)d\xi, \qquad (2.16)$$

where

$$G_{\xi x}(x,y;\;0,\eta)=\frac{1}{2\sqrt{\pi(y-\eta)}}\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}\left[\frac{1}{y-\eta}-\frac{(x+2n)^2}{2(y-\eta)^2}\right]e^{-\frac{(x+2n)^2}{4(y-\eta)}}=$$

$$=\frac{d}{d\eta}\left[\frac{1}{\sqrt{\pi(y-\eta)}}e^{\frac{x^2}{4(y-\eta)}}+\frac{1}{\sqrt{\pi(y-\eta)}}\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}e^{-\frac{(x+2n)^2}{4(y-\eta)^2}}\right], \qquad (2.17)$$

$$G_{\xi x}(x,y;\,1,\eta)=\frac{1}{2\sqrt{\pi(y-\eta)}}\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}\left\{\left(\frac{1}{2(y-\eta)}-\frac{(x-1+2n)^2}{4(y-\eta)^2}\right)e^{-\frac{(x-1+2n)^2}{4(y-\eta)}}+\right.$$

$$\left.+\left(\frac{1}{2(y-\eta)}-\frac{(x+1+2n)^2}{4(y-\eta)^2}\right)e^{-\frac{(x+1+2n)^2}{4(y-\eta)}}\right\}=$$

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$$=\frac{d}{d\eta}\left[\frac{1}{2\sqrt{\pi(y-\eta)}}e^{-\frac{(x-1)^2}{4(y-\eta)}}+\frac{1}{2\sqrt{\pi(y-\eta)}}\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}e^{-\frac{(x-1+2n)^2}{4(y-\eta)}}\right]+$$

$$+\frac{d}{d\eta}\left[\frac{1}{2\sqrt{\pi(y-\eta)}}e^{-\frac{(x+1)^2}{4(y-\eta)}}+\frac{1}{2\sqrt{\pi(y-\eta)}}\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}e^{-\frac{(x+1+2n)^2}{4(y-\eta)}}\right], \quad (2.18)$$

$$G_x(x,y;\;\xi,0)=\frac{1}{2\sqrt{\pi y}}\sum_{n=-\infty}^{+\infty}e^{-\frac{(x+2n)^2+\xi^2}{4y}}\left[\frac{\xi}{y}ch2\xi(x+2n)-\frac{x+n}{y}sh2\xi(x+2n)\right] \quad (2.19).$$

Using the formula (2.16) and making integration by parts, taking into account (2.17), (2.18) and (2.6), owing to $\lim\limits_{z\to 0} z^{-\sigma}e^{-1/z}=0, \quad (\sigma>0)$, we have

$$u_x(x,y)=-\frac{1}{\sqrt{\pi}}\int_0^y\frac{\tau_2'(\eta)}{\sqrt{y-\eta}}e^{-\frac{x^2}{4(y-\eta)}}d\eta-\frac{1}{\sqrt{\pi}}\int_0^y\frac{\tau_2'(\eta)}{\sqrt{y-\eta}}\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}e^{-\frac{(x+2n)^2}{4(y-\eta)}}d\eta+$$

$$+\frac{1}{2\sqrt{\pi}}\int\limits_0^y \frac{\varphi'_1(\eta)}{\sqrt{y-\eta}}e^{-\frac{(x-1)^2}{4(y-\eta)}}\,d\eta+\frac{1}{2\sqrt{\pi}}\int\limits_0^y \frac{\varphi'_1(\eta)}{\sqrt{y-\eta}}\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty} e^{-\frac{(x-1+2n)^2}{4(y-\eta)}}\,d\eta+$$

$$+\frac{1}{2\sqrt{\pi}}\int\limits_0^y \frac{\varphi'_1(\eta)}{\sqrt{y-\eta}}e^{-\frac{(x+1)^2}{4(y-\eta)}}\,d\eta+\frac{1}{2\sqrt{\pi}}\int\limits_0^y \frac{\varphi'_1(\eta)}{\sqrt{y-\eta}}\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty} e^{-\frac{(x+1+2n)^2}{4(y-\eta)}}\,d\eta+$$

$$+\frac{1}{2\sqrt{\pi y}}\int\limits_0^1 \sum_{n=-\infty}^{+\infty} e^{-\frac{(x+2n)^2+\xi^2}{4y}}\left[\frac{\xi}{y}ch2\xi(x+2n)-\frac{x+n}{y}sh2\xi(x+2n)\right]\tau_1(\xi)\,d\xi. \tag{2.20}$$

According to the conditions of the problem, passing to the limit as $x \longrightarrow +0$ in (2.20) considering (4), (2.2) and the identities:

$$\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty} e^{-\frac{(2n-1)^2}{4(y-\eta)}}=\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty} e^{-\frac{(2n+1)^2}{4(y-\eta)}}=e^{-\frac{1}{4(y-\eta)}}+2\sum_{n=1}^{+\infty} e^{-\frac{(2n+1)^2}{4(y-\eta)}},$$

$$\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty} e^{-\frac{(2n)^2}{4(y-\eta)}}=2\sum_{n=1}^{+\infty} e^{-\frac{n^2}{y-\eta}},$$

we obtain the functional relation between $\tau_2(y)$ and $\nu_2(y)$, brought from the domain D_1 by I :

$$\nu_2(y)=-\frac{1}{\sqrt{\pi}}\int\limits_0^y \frac{\tau_2'(\eta)}{\sqrt{y-\eta}}d\eta-\frac{1}{\sqrt{\pi}}\int\limits_0^y \frac{K_1(y,\eta)}{\sqrt{y-\eta}}\tau'_2(\eta)\,d\eta+F_1(y,\varphi'_1,\tau_1), \tag{2.21}$$

where

$$K_1(y,\eta)=2\sum_{n=1}^{+\infty} e^{-\frac{n^2}{y-\eta}}, \tag{2.22}$$

$$F_1(y,\varphi'_1,\tau_1)=\frac{2}{\sqrt{\pi}}\int\limits_0^y \frac{\varphi'_1(\eta)}{\sqrt{y-\eta}}e^{-\frac{1}{4(y-\eta)}}\,d\eta+\frac{2}{\sqrt{\pi}}\int\limits_0^y \frac{\varphi'_1(\eta)}{\sqrt{y-\eta}}\sum_{n=1}^{+\infty} e^{-\frac{(1+2n)^2}{4(y-\eta)}}\,d\eta+$$

$$+\frac{1}{2\sqrt{\pi y}}\int\limits_0^1 \sum_{n=-\infty}^{+\infty} e^{-\frac{4n^2+t^2}{4y}}\left[\frac{t}{y}ch4\,t\,n-\frac{n}{y}sh4\,t\,n\right]\tau_1(t)\,dt. \tag{2.23}$$

3 Investigation of the BS-problem

The following theorem is proved.

Theorem 3.1. *If conditions (1.10) - (1.12) are satisfied, then in the domain D there exists a unique regular solution of the BS-problem.*

Proof. Excluding $v_1(x)$ from the relations (2.9), (2.10), (2.14) owing to the gluing condition (1.4) and conditions (1.5), (1.6), (1.8), $u|_{x=0} = \tau_2(y)$ considering also (1.10), (2.1), (2.6) we obtain following problems:

$$\tau_1''(x) - d_1(x)\tau_1'(x) - d'_1(x)\tau_1(x) = -2c'_1(x), (x,0) \in J_1, \quad (3.1)$$

$$\tau_1(0) \equiv \tau_2(0) = \qquad \tau_1(k_1) = 0 \quad (3.2)$$

and$\phi_3(0) = 0$,

$$\tau''_1(x) - \tau'_1(x) = -\frac{2}{1-\mu_1}\delta'_1(x), \quad (x,0) \in J_2, \quad (3.3)$$

$$\tau_1 \qquad \tau_1(1) = \phi_1(0). \quad (3.4)$$

$(k_1) = 0$,

The solution of (3.1) satisfying the first conditions (3.2) can be an equivalent way reduced in to the Volterra integral equation of the second kind with respect to $\tau_1'(x)$:

$$\tau_1'(x) - \int_0^x M_1(x,t)\tau_1'(t)\,dt = \Phi_1(x), \quad (x,0) \in \bar{J}_1, \quad (3.5)$$

where

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$M_1(x,t) = d_1(t) + d_1(z)dz, \Phi_1(x) = -2c_1(x) + \tau^0{}_1(0).$

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From this, by virtue of (1.12), we conclude that

$\Phi_1(x) \in C\,[0, k_1]$\\$C^2(0, k_1), M_1(x,t) \in C\,([0,k_1] \times [0,k_1])$. (3.6)

According to the theory of Volterra type integral equations of the second kind, we conclude that the integral equation (3.5) is uniquely solvable in the class $C\,[0,k_1] \cap C^2(0,k_1)$ and its solution is given by the formula

$$\tau_1'(x) = \Phi_1(x) + \int_0^x \tilde{M}_1(x,t)\Phi_1(t)dt, \quad (x,0) \in \bar{J}_1, \quad (3.7)$$

where $\tilde{M}_1(x,t)$ is resolvent- kernel of $M_1(x,t)$.

Integrating (3.7) from 0 to x considering $\tau_1(0) = 0$, we have

$$\tau_1(x) = \int_0^x \Phi_1(t)dt + \int_0^x dt \int_0^t \tilde{M}_1(t,z)\Phi_1(z)dz, \quad (x,0) \in \bar{J}_1, \qquad (3.8)$$

Based on (3.6), from (3.8) we conclude that

$$\tau_1(x) \in C^1(\bar{J}_1) \bigcap C^2(J_1). \qquad (3.9)$$

Now, putting in (3.8) $x = k_1$ owing to $\tau_1(k_1) = 0$ and the form of the function $\Phi_1(x)$, we find an unknown constant $\tau^0{}_1(0)$:

$$\tau'_1(0) = \frac{2\left[\int_0^{k_1} c_1(t)dt + \int_0^{k_1} dt \int_0^t \tilde{M}_1(t,z)c_1(z)\,dz\right]}{k_1 + \int_0^{k_1} dt \int_0^t \tilde{M}_1(t,z)dz}. \qquad (3.10)$$

Based on (1.11), it follows that the resolvent-kernel is also positive, i.e. $\tilde{M}_1(x,t) > 0,\ \forall\, x,t \in [0,k_1]$. Hence, the denominator of formula

(3.10) for any $0 \le x \le k_1, 0 \le t \le k_1$ does not vanish, that is

$$k_1 + \int_0^{k_1} dt \int_0^t \tilde{M}_1(t,z)dz > 0.$$

Solving the problems (3.3) and (3.4), we represent in the form

$$\tau_1(x) = c_0(e^x - e^{k_1}) - \frac{2}{1-\mu_1}\left[\int_{k_1}^x e^{x-t}\delta'_1(t)\,dt - \delta_1(x) + \delta_1(k_1)\right], (x,0) \in \bar{J}_2 \qquad (3.11),$$

where

$$c_0 = \frac{\frac{2}{1-\mu_1}\left[\int_{k_1}^1 e^{1-t}\delta'_1(t)\,dt - \delta_1(1) + \delta_1(k_1)\right] + \varphi_1(0)}{e - e^{k_1}}.$$

By virtue (1.12), from (3.11) we conclude that

$$\tau_1(x) \in C^1(\bar{J}_2) \cap C^2(J_2). \qquad (3.12)$$

Supplying (3.8) and (3.11) into (2.9) and (2.10) respectively, considering (1.12), (1.13), (3.9), (3.12) we define the function $\nu_1(x)$ from the class

$$\nu_1(x) \in C(\bar{J}_1) \cap C^1(J_1) \ \ and\ \nu_1(x) \in C(\bar{J}_2) \cap C^1(J_2). \qquad (3.13)$$

Eliminating $\nu_2(y)$ and considering (4), (1.11), (2.6), from (2.12), (2.21) and (2.13), (2.21) respectively, we obtain the integral equation with respect to $\tau'_2(y)$:

$$\tau'_2(y) + \int_0^y K_2(y,\eta)\tau'_2(\eta)\,d\eta = F_2(y), (0,y) \in I_1 \qquad (3.14)$$

and

$$\tau_2'(y) + \int_0^y K_3(y,\eta)\tau_2'(\eta)\,d\eta = F_3(y), (0,y) \in I_2 \quad , \qquad (3.15)$$

where

$$K_2(y,t) = \frac{d'_2(y)}{d_2(y)} + \frac{1 + K_1(y,t)}{d_2(y)\cdot\sqrt{\pi(y-t)}} \quad , \qquad (3.16)$$

$$(3.17) \quad K_3(y,t) = (1 + K_1(y,t))/\sqrt{\pi(y-t)},$$

$$F_2(y) = [2c^0{}_2(y) - F_1(y,\phi^0{}_1,\tau_1)]/d_2(y) \qquad (3.18)$$

$$F_3(y) = 2\delta^0{}_2(y)/(1-\mu_2) + F_1(y,\phi^0{}_1,\tau_1). \qquad (3.19)$$

Based on $\lim_{z\to 0} z^{-\sigma}e^{-1/z} = 0$ for any fixed $\sigma > 0$, considering (1.11), (1.12), (1.13), (3.9), (3.12) we conclude that

1) $K_2(y,t)$ is continuously in $\{(y,t) : 0 \le t < y \le k_2\}$ and with $y \to t$ admits an estimate

$$|K_2(y,t)| \le const(y-t)^{-\frac{1}{2}}; \qquad (3.20)$$

2) $K_3(y,t)$ is continuously in $\{(y,t) : k_2 \le t \le y \le h\}$ and with $y \to t$ admits an estimate

$$|K_3(y,t)| \le const(y-t)^{-\frac{1}{2}}; \qquad (3.21)$$

3)

$F_2(y) \in C\,[0,k_2] \cap C^2\,(0,k_2)$ *and* $F_3(y) \in C\,[k_2,h] \cap C^2\,(k_2,h)$. (3.22)

Thus, taking (3.20), (3.21) and (3.22) into account, equation (3.14) and (3.15) are Volterra type integral equations of the second kind with a weak singularity.

According to the theory of Volterra type integral equations of the second kind [10], we conclude that the integral equations (3.14) and (3.15) are uniquely solvable in the class $C\,[0,k_2]\cap C^2\,(0,k_2)$ and $C\,[k_2,h]\cap C^2\,(k_2,h)$, respectively, and their solution is given by

$$\tau'_2(y) = F_2(y) - \int_0^y \tilde{K}_2(y,t)F_2(t)\,dt, (0,y) \in \bar{I}_1 \qquad (3.23)$$

and

$$\tau'_2(y) = F_3(y) - \int_0^y \tilde{K}_3(y,t)F_2(t)\,dt, (0,y) \in \bar{I}_2 \quad , \qquad (3.24)$$

where $\tilde{K}_j(y,t)$ resolvent- kernel of $K_j(y,t)(j = 2,3)$.

Using by $\tau_2(0) = 0$, $\tau_2(k_2) = 0$ from (3.23) and (3.24) we find the function $\tau_2(y)$:

$$\tau_2(y) = \int_0^y \left\{ F_2(t) - \int_0^t \tilde{K}_2(t,z)F_2(z)dz \right\} dt, (0,y) \in \bar{I}_1 \qquad (3.25)$$

and

$$\tau_2(y) = \int_{k_2}^y \left\{ F_3(t) - \int_0^t \tilde{K}_3(t,z)F_3(z)dz \right\} dt, (0,y) \in \bar{I}_2 \qquad (3.26)$$

and it belongs to the class

$$\tau_2(y) \in C^1[0,k_2] \cap C^2(0,k_2) \textit{ and } \tau_2(y) \in C^1[k_2,h] \cap C^2(k_2,h). \qquad (3.27)$$

Substituting (3.27) into (2.12) and (2.13) owing to (4), (1.12), (1.13), (2.2), (3.25), (3.26) we define that the function $\nu_2(y)$ from the class

$\nu_2(y) \in C[0,k_1] \cap C^1(0,k_1)$ *and* $\nu_2(y) \in C[k_2,h] \cap C^1(k_2,h)$. (3.28)

Thus, the solution of the BS-problem can be restored in domain D_1 as a solution of the first boundary-value problem for equation (1.1) (see (2.15)),

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and in domains D_j $(j = 2,3)$ as a solution of the Cauchy problem for equation (1.1).

Thus, the BS-problem is uniquely solvable.

The theorem is proved. References

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